MATHEMATICS OF COMPUTATION
VOLUME 62, NUMBER 206
APRIL 1994, PAGES 719–724

ON A THEOREM OF C. POSSE CONCERNING GAUSSIAN QUADRATURE OF CHEBYSHEV TYPE

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ABSTRACT. We consider (n, m) Chebyshev formulae of algebraic degree m using n nodes. The aim of this short note is to show that by a simple algebraic method C. Posse's theorem concerning Gaussian quadrature of Chebyshev type can be improved. Furthermore, we given an application of this method to Gauss-Kronrod quadrature of Chebyshev type.

1. INTRODUCTION AND STATEMENT OF THE MAIN RESULT

Let L be a (bounded) linear functional on $C[a, b]$. We say that L admits an (n, m) Chebyshev formula if there are n nodes $x_{\nu,n}^C \in \mathbb{R}$ such that

$$L[p_\mu] = \frac{L[p_0]}{n} \sum_{\nu=1}^{n} (x_{\nu,n}^C)^\mu \quad \text{for } \mu = 0, 1, \dots, m, \tag{1}$$

where, here and in the following, p_μ denotes the monomial $p_\mu(x) = x^\mu$.

Chebyshev formulae have been investigated for more than a hundred years (see, e.g., [6] and the references cited therein). They were first considered by Chebyshev [3], who showed that the linear functionals $T_{\alpha,\beta,\gamma}$,

$$T_{\alpha,\beta,\gamma}[f] := \alpha \int_\beta^\gamma \frac{f(x)}{\sqrt{|x-\beta|\,|x-\gamma|}}\,dx, \qquad \alpha, \beta, \gamma \in \mathbb{R}, \tag{2}$$

admit $(n, 2n-1)$ Chebyshev formulae for each $n \in \mathbb{N}$, i.e., that each Gaussian quadrature formula is of Chebyshev type. Let us additionally note that the linear functionals $S_{\eta,\xi}$,

$$S_{\eta,\xi}[f] := \eta f(\xi), \qquad \eta, \xi \in \mathbb{R}, \tag{3}$$

trivially admit (n, m) Chebyshev formulae for all $n, m \in \mathbb{N}$. By a result of Posse [11], $T_{\alpha,\beta,\gamma}$ and $S_{\eta,\xi}$ are the only linear functionals on $C[a, b]$ admitting $(n, 2n-1)$ Chebyshev formulae for each $n \in \mathbb{N}$. Recently, using methods of complex analysis and Faber polynomials, Peherstorfer [8] has proved the surprising result, that $T_{\alpha,\beta,\gamma}$ and $S_{\eta,\xi}$ also are the only (positive) linear functionals on $C[a, b]$, admitting $(1, 1)$ and $(n, n+1)$ Chebyshev formulae for each $n \in \mathbb{N}\backslash\{1\}$. For other improvements of Posse's result see, e.g., [6, 5, 9].

Received by the editor November 6, 1992 and, in revised form, March 24, 1993.
1991 *Mathematics Subject Classification.* Primary 65D30, 41A55.
Key words and phrases. Chebyshev quadrature, Gaussian quadrature, Gauss-Kronrod quadrature.

The aim of this note is to show that by a simple algebraic method, introduced by Radau [12] and extended, e.g., in [2, 1, 4], we can obtain more general results. This method is based on Newton's identities (see, e.g., [10, p. 150 ff]), which, for n arbitrary complex numbers z_ν, yield

$$(4)\qquad \begin{cases} s_1 + a_1 = 0, \\ s_2 + a_1 s_1 + 2a_2 = 0, \\ \cdots\cdots \\ s_{n-1} + a_1 s_{n-2} + \cdots + a_{n-2} s_1 + (n-1)a_{n-1} = 0, \\ s_{\lambda+n} + a_1 s_{\lambda+n-1} + \cdots + a_{n-1} s_{\lambda+1} + a_n s_\lambda = 0 \qquad (\lambda = 0, 1, 2, \dots), \end{cases}$$

where

$$(5)\qquad \begin{aligned} s_\mu &= \sum_{\nu=1}^{n} z_\nu^\mu \qquad (\mu = 0, 1, 2, \dots), \\ F(t) &= (t - z_1)(t - z_2)\cdots(t - z_n) = t^n + a_1 t^{n-1} + \cdots + a_{n-1} t + a_n. \end{aligned}$$

Theorem. *Let $(n_i)_{i=1}^\infty$ be a strictly increasing sequence of natural numbers. Let L and H be linear functionals on $C[a, b]$, both admitting (n_i, n_{i+1}) Chebyshev formulae for each $i \in \mathbb{N}$. If*

$$(6)\qquad L[p_\mu] = H[p_\mu] \quad \text{for } \mu = 0, 1, \dots, n_1,$$

then the identity $L = H$ follows.

Proof. First, let $L[p_0] \neq 0$. Applying Newton's identities to (1), we get for $i = 1, 2, \dots$

$$(7)\qquad \sum_{\nu=1}^{n_i} (x_{\nu, n_i}^C)^\mu = \frac{n_i}{L[p_0]} L[p_\mu] \quad \text{for } \mu = 0, 1, \dots, n_{i+1}.$$

For given $L[p_0], L[p_1], \dots, L[p_{n_i}]$ we directly obtain that the values of $L[p_\mu]$ are uniquely determined for each $\mu \in \{0, 1, 2, \dots, n_{i+1}\}$. Using (6), we have $L[p_\mu] = H[p_\mu]$ for each $\mu \in \mathbb{N}_0$. Since L and H are bounded, by the approximation theorem of Weierstrass the result follows. If $L[p_0] = 0$, then with (1) we have $L = H \equiv 0$. □

The following corollary extends the results on Chebyshev formulae mentioned in the introduction.

Corollary 1. *Let $(n_i)_{i=1}^\infty$ be a strictly increasing sequence of natural numbers. Let L be a linear functional on $C[a, b]$ admitting (n_i, n_{i+1}) Chebyshev formulae for each $i \in \mathbb{N}$. Then there exist $\eta, \xi \in \mathbb{R}$ or $\alpha, \beta, \gamma \in \mathbb{R}$, such that, using the notations in* (2) *and* (3),

(i) $L = S_{\eta,\xi}$ *if* $n_1 = 1$,

(ii) $L = S_{\eta,\xi}$ *or* $L = T_{\alpha,\beta,\gamma}$ *if* $n_1 = 2$ *and* $n_{i+1} < 2n_i$ *for each* $i \in \mathbb{N}$.

Proof. As in the proof of the theorem, we can assume that $L[p_0] \neq 0$. First, let $L[p_2]L[p_0] = L^2[p_1]$. We define

$$(8)\qquad H = S_{\eta,\xi}, \quad \eta = L[p_0], \quad \xi = L[p_1]/L[p_0].$$

We have $L[p_\mu] = H[p_\mu]$ for $\mu = 0, 1, 2$; therefore, the result follows from the theorem. Now let $L[p_2]L[p_0] > L^2[p_1]$. We define

$$\overline{H} = T_{\alpha,\beta,\gamma}, \quad \alpha = \frac{L[p_0]}{\pi}, \quad \beta = \frac{L[p_1] - \delta}{L[p_0]}, \quad \gamma = \frac{L[p_1] + \delta}{L[p_0]}, \quad \delta = [2(L[p_2]L[p_0] - L^2[p_1])]^{1/2}. \tag{9}$$

A short calculation, using

$$\overline{H}[f] = \alpha \int_{-1}^{1} f\left(\frac{\gamma - \beta}{2} y + \frac{\gamma + \beta}{2}\right)(1 - y^2)^{-1/2}\, dy, \tag{10}$$

gives $L[p_\mu] = \overline{H}[p_\mu]$ for $\mu = 0, 1, 2$. Since $\overline{H}$ admits $(n, 2n-1)$ Chebyshev formulae for each $n \in \mathbb{N}$, the result follows from the theorem. Finally, let $L[p_2]L[p_0] < L^2[p_1]$. This inequality is equivalent to the inequality

$$(x_{1,2}^C)^2 + (x_{2,2}^C)^2 < (x_{1,2}^C + x_{2,2}^C)^2/2, \tag{11}$$

which is impossible for real numbers. □

Remarks. 1. We say that a linear functional L on $C[a, b]$ admits extended (n, m) Chebyshev formulae if $x_{\nu,n}^C \in \mathbb{C}$, $x_{\nu,n}^C$ real or complex conjugate, and (1) holds. Newton's identities (4) are valid for $z_\nu \in \mathbb{C}$, and therefore the theorem is also valid for extended Chebyshev formulae.

2. In the proof of Corollary 1, we see that (11) is possible if and only if $x_{1,2}^C, x_{2,2}^C \in \mathbb{C}$, $x_{1,2}^C = u + iv$, $x_{2,2}^C = u - iv$, and $v \neq 0$. We obtain $L[p_1] = uL[p_0]$, $L[p_2] = (u^2 - v^2)L[p_0]$. Defining $\tilde{f} := (p_1 - up_0)^2 + v^2 p_0/2$, we have

$$L[\tilde{f}] = -\frac{1}{2}v^2 L[p_0], \qquad \tilde{f} > 0, \tag{12}$$

which is impossible if $L[p_0]L$ is a positive functional. Therefore, for positive functionals L, Corollary 1 also is valid for extended Chebyshev formulae. For $n_i = i + 1$ this has been proved in [8].

3. In Corollary 1(ii) the assumption $n_{i+1} < 2n_i$ cannot be omitted: consider the functional $\overline{S}_{\eta,\xi_1,\xi_2}$ defined by $\overline{S}_{\eta,\xi_1,\xi_2}[f] = \frac{\eta}{2}[f(\xi_1) + f(\xi_2)]$, which admits $(2n, m)$ Chebyshev formulae for all $n, m \in \mathbb{N}$.

4. In Corollary 1, we only have considered $n_1 = 1$ or $n_1 = 2$. For $n_1 > 2$, using the method described, we are also able to investigate linear functionals admitting (extended) Chebyshev formulae. The author intends to state such results and further applications in a forthcoming paper.

2. Application to Chebyshev formulae having preassigned nodes

The above method is also helpful if some of the nodes $x_{\nu,n}^C$ of the Chebyshev formulae are preassigned. As an example, we assume that the linear functional L also admits a Gaussian formula of order k; i.e., there exist k nodes $x_{\nu,k}^G$ and k weights $a_{\nu,k}^G \in \mathbb{R}$ such that

$$L[p_\mu] = \sum_{\nu=1}^{k} a_{\nu,k}^G (x_{\nu,k}^G)^\mu \quad \text{for } \mu = 0, 1, \ldots, 2k-1. \tag{13}$$

If $L[p_0] \neq 0$, then a Gaussian formula of order 1 trivially always exists and is uniquely determined by

$$x^G_{1,1} = L[p_1]/L[p_0]. \tag{14}$$

If there exists also a $(3, 2)$ Chebyshev formula such that $x^G_{1,1}$ is one of its nodes, then a short calculation using (13) and (1) shows that

$$\begin{gathered} x^C_{1,3} := x^G_{1,1}, \quad x^C_{2,3} = x^C_{1,3} - \sqrt{3/2}\psi, \quad x^C_{3,3} = x^C_{1,3} + \sqrt{3/2}\psi, \\ \psi = \frac{1}{L[p_0]}\sqrt{L[p_0]L[p_2] - L^2[p_1]}. \end{gathered} \tag{15}$$

Therefore, it is necessary and sufficient for the existence of such a $(3, 2)$ Chebyshev formula that

$$L[p_0]L[p_2] \geq L^2[p_1]. \tag{16}$$

Furthermore, if this $(3, 2)$ Chebyshev formula is also a $(3, 3)$ Chebyshev formula, then it follows that

$$L[p_3] = \frac{L[p_1]}{L^2[p_0]}(3L[p_0]L[p_2] - 2L^2[p_1]). \tag{17}$$

Inequality (16) is equivalent to the existence of a $(2, 2)$ Chebyshev formula,

$$x^C_{1,2} = x^G_{1,1} - \psi, \qquad x^C_{2,2} = x^G_{1,1} + \psi, \tag{18}$$

where ψ is defined in (15). A further short calculation shows that this $(2, 2)$ Chebyshev formula is also a $(2, 3)$ Chebyshev formula, i.e., a Gaussian formula of order 2, if and only if (17) additionally holds. Therefore, the existence of a $(3, 3)$ Chebyshev formula using the node $x^G_{1,1}$ is equivalent to the existence of a $(2, 3)$ Chebyshev formula.

Now, one may ask, e.g., if there exists an (n, m) Chebyshev formula having some of the nodes $x^G_{\nu,k}$ of a Gaussian formula of order k. In this situation, using methods from the theory of orthogonal polynomials, Notaris [7], for positive L, has recently proved the following interesting result: If, for each $n \in \mathbb{N}$, there exists a $(2n + 1, 3n + 1)$ Chebyshev formula with $\{x^G_{\nu,n} | \nu = 1, 2, \dots, n\} \subseteq \{x^C_{\nu,2n+1} | \nu = 1, 2, \dots, 2n + 1\}$—i.e., these Chebyshev formulae are so-called Gauss-Kronrod formulae—then L is of type $S_{\eta,\xi}$ being defined in (3). The following Corollary 2 extends this result.

Corollary 2. *Let L be a linear functional admitting a $(3, 4)$ Chebyshev formula and $(2n+1, 2n+3)$ Chebyshev formulae for each $n \in \mathbb{N}\backslash\{1\}$. For $L[p_0] \neq 0$, let the node $x^G_{1,1}$ of the Gaussian formula of order 1 be a node of the $(3, 4)$ Chebyshev formula, and let the two nodes $x^G_{1,2}$, $x^G_{2,2}$ of the Gaussian formula of order 2 be nodes of the $(5, 7)$ Chebyshev formula. Then, there exist $\eta, \xi \in \mathbb{R}$ such that $L = S_{\eta,\xi}$.*

Proof. Let $L[p_0] \neq 0$. The existence of a $(3, 3)$ Chebyshev formula using the node $x^G_{1,1}$ implies that the Gaussian formula of order 2 is a $(2, 3)$ Chebyshev formula; see (17) and (18) above. If there exists a $(5, 3)$ Chebyshev formula having nodes $x^C_{1,5} := x^G_{1,2} = x^C_{1,2}$ and $x^C_{2,5} := x^G_{2,2} = x^C_{2,2}$, then a short calculation using

$$(19)\qquad \begin{aligned} L[p_\mu] &= \frac{L[p_0]}{5}\left\{\sum_{\nu=1}^{2}(x^C_{\nu,2})^\mu + \sum_{\nu=3}^{5}(x^C_{\nu,5})^\mu\right\} \\ &= \frac{L[p_0]}{5}\left\{2\frac{L[p_\mu]}{L[p_0]} + \sum_{\nu=3}^{5}(x^C_{\nu,5})^\mu\right\} \quad \text{for } \mu = 0, 1, 2, 3 \end{aligned}$$

shows that $\{x^C_{3,5}, x^C_{4,5}, x^C_{5,5}\} = \{x^C_{1,3}, x^C_{2,3}, x^C_{3,3}\}$, and therefore, since this $(5, 3)$ Chebyshev formula is also a $(5, 4)$ Chebyshev formula, it follows that

$$(20)\qquad L[p_4] = \frac{L[p_0]}{5}\left\{\sum_{\nu=1}^{3}(x^C_{\nu,3})^4 + \sum_{\nu=1}^{2}(x^C_{\nu,2})^4\right\}.$$

On the other hand, the existence of a $(3, 4)$ Chebyshev formula having a node $x^C_{1,3} = x^G_{1,1}$ yields the identity

$$(21)\qquad L[p_4] = \frac{L[p_0]}{3}\sum_{\nu=1}^{3}(x^C_{\nu,3})^4.$$

By comparing (20) and (21), it follows from(15) and (18) that

$$(22)\qquad L^2[p_0]\psi^2 = L[p_0]L[p_2] - L^2[p_1] = 0.$$

Therefore, all nodes $x^C_{\nu,3}$ and $x^C_{\nu,5}$ are equal to $x^G_{1,1}$. Since the $(5, 4)$ Chebyshev formula is also a $(5, 7)$ Chebyshev formula, we have $L[p_\mu] = S_{\eta,\xi}[p_\mu]$ for $\mu = 0, 1, \dots, 7$, where η and ξ are given in (8). The result now follows from the theorem. If $L[p_0] = 0$, then with (1) we have $L \equiv 0$, which is fulfilled for $S_{\eta,\xi}$ with $\eta = 0$. □

By Corollary 1 it follows that, if there exists a $(2, 3)$ Chebyshev formula and $(2n + 1, 2n + 3)$ Chebyshev formulae for each $n \in \mathbb{N}$, then L is of type $S_{\eta,\xi}$ or $T_{\alpha,\beta,\gamma}$. For $(2n + 1, 2n + 2)$ Chebyshev formulae, using the above method, we have the following result.

Corollary 3. *Let L be a linear functional admitting $(2n+1, 2n+2)$ Chebyshev formulae for each $n \in \mathbb{N}$. For $L[p_0] \neq 0$ let $x^G_{1,1} = L[p_1]/L[p_0]$ be a node of each of these $(2n + 1, 2n + 2)$ Chebyshev formulae. Then there exist $\eta, \xi \in \mathbb{R}$ or $\alpha, \beta, \gamma \in \mathbb{R}$ such that $L = S_{\eta,\xi}$ or $L = T_{\alpha,\beta,\gamma}$.*

Proof. Let $L[p_0] \neq 0$. Since L admits a $(3, 2)$ Chebyshev formula having the node $x^C_{1,3} := x^G_{1,1}$, the relations (15) imply that $L[p_0]L[p_2] \geq L^2[p_1]$. Therefore, there exists a functional H of type $S_{\eta,\xi}$ or $T_{\alpha,\beta,\gamma}$ such that $H[p_\mu] = L[p_\mu]$ for $\mu = 0, 1, 2$—see the proof of Corollary 1. Now assume that, for given $n \in \mathbb{N}$,

$$(23)\qquad H[p_\mu] = L[p_\mu] \quad \text{for } \mu = 0, 1, \dots, 2n.$$

In the following, by $x^{CH}_{\nu,n}$ we denote the nodes of the $(n, 2n - 1)$ Chebyshev formula of H. Using (1), we have

$$(24)\qquad \sum_{\nu=1}^{2n+1}(x^C_{\nu,2n+1})^\mu = (2n + 1)\frac{L[p_\mu]}{L[p_0]} \quad \text{for } \mu = 0, 1, \dots, 2n + 2.$$

From (23) it follows that the nodes $x^C_{1,2n+1}, x^C_{2,2n+1}, \ldots, x^C_{2n+1,2n+1}$ are also the nodes of a $(2n+1, 2n)$ Chebyshev formula of H. Let F^C_{2n+1} and F^{CH}_{2n+1} be the polynomials

$$(25) \qquad F^C_{2n+1}(t) = \prod_{\nu=1}^{2n+1} (t - x^C_{\nu,2n+1}), \qquad F^{CH}_{2n+1}(t) = \prod_{\nu=1}^{2n+1} (t - x^{CH}_{\nu,2n+1}).$$

Newton's identities (4) and (5) now show that $F^C_{2n+1}(t) - F^{CH}_{2n+1}(t)$ is equal to a fixed constant c for all $t \in \mathbb{R}$. Since $x^G_{1,1}$ is a zero of F^{CH}_{2n+1} and since $x^G_{1,1}$ is also a zero of F^C_{2n+1}, it follows that $F^C_{2n+1} \equiv F^{CH}_{2n+1}$. Therefore, (24) yields $H[p_\mu] = L[p_\mu]$ for $\mu = 0, 1, 2, \ldots, 2n+2$. Now, by induction, we have $H[p_\mu] = L[p_\mu]$ for each $\mu \in \mathbb{N}$, which, using the approximation theorem of Weierstrass, yields $L = H$. □

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